HIGHER DIMENSIONAL BELL-SZEKERES METRIC

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The collision of pure electromagnetic plane waves with collinear polarization in N-dimensional ($N = 2 + n$) Einstein-Maxwell theory is considered. A class of exact solutions for the higher dimensional Bell-Szekeres metric is obtained and its singularity structure is examined.

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I. INTRODUCTION

One of the main fields of interest in general relativity is the collision of gravitational plane waves. Colliding plane wave space-times have been investigated in detail in general relativity [1]. The first exact solution of the Einstein-Maxwell equations representing colliding plane shock electromagnetic waves with collinear polarizations was obtained by Bell and Szekeres (BS) [2]. This solution is conformally flat in the interaction region and its singularity structure has been considered by Matzner and Tipler [3], Clark and Hayward [4] and Helliwell and Konkowski [5]. Later Halil [6], Gürses and Halil [7], Griffiths [8] and Chandrasekhar and Xanthhopolos [9] studied exact solutions of the Einstein-Maxwell equations describing the collision of gravitational and electromagnetic waves. Furthermore Gürses and Sermutlu [10] , more recently Halil and Sakalli [11] have obtained the extensions of the BS solution in the Einstein-Maxwell-dilaton and Einstein-Maxwell-axion theories, respectively. Recently, we have given a higher even dimensional extensions of the vacuum colliding gravitational waves with collinear and also combinations with noncollinear polarizations. [12, 13].

In this work we give a higher dimensional formulation of BS metric. We present an exact solution generalizing the BS solution and examine the singularity structure of the corresponding spacetimes in the context of curvature and Maxwell invariants. We show that this space-time, unlike BS metric, is not conformally flat.

In Sec. II, we give a brief review of the BS solution and in Sec. III, we formulate the N dimensional Einstein-Maxwell equations. In Sec. IV, we present the N dimensional colliding exact plane wave solutions describing the collision of shock electromagnetic waves. We also examine the singularity structure of the corresponding spacetimes and show that interaction region of our solution admits curvature singularities.

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II. THE BELL-SZEKERES METRIC

The BS metric is given by

$$ds^2 = 2dudv + e^{-U}(e^V dx^2 + e^{-V} dy^2) \tag{1}$$

where the metric functions U and V depend on the null coordinates u and v. The electromagnetic vector potential has a single nonzero component $A = (0, 0, 0, A)$, where A is functions of u and v. The complete solution of the Einstein-Maxwell equations is

$$\begin{aligned} U &= -\log(f(u) + g(v)), \quad A = \gamma(pw - rq), \\ V &= \log(rw - pq) - \log(rw + pq), \end{aligned} \tag{2}$$

where

$$\begin{aligned} r &= (\frac{1}{2} + f)^{1/2}, \qquad p = (\frac{1}{2} - f)^{1/2}, \\ w &= (\frac{1}{2} + g)^{1/2}, \qquad q = (\frac{1}{2} - g)^{1/2} \end{aligned} \tag{3}$$

with

$$f = \frac{1}{2} - \sin^2 P, \qquad g = \frac{1}{2} - \sin^2 Q. \tag{4}$$

Here $P = au\Theta(u)$, $Q = bv\Theta(v)$, where Θ is the Heaviside unit step function, a and b are arbitrary constants and $\gamma^2 = \frac{8\pi}{\kappa}$ with κ being the gravitational constant. The nature of the space-time singularity can be extracted from the curvature invariant which is a finite quantity

$$I = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \tag{5}$$

$$\begin{aligned} &= \frac{8}{w^2p^2q^2r^2(f+g)^2}[fgf_u^2g_v^2 + w^2p^2q^2r^2 f_{uu}g_{vv} \\ &+ \frac{1}{4}((3g-f)r^2p^2 f_{uu}g_v^2 + (3f-g)w^2q^2 g_{vv} f_u^2)] \\ &= 32a^2b^2. \end{aligned} \tag{6}$$

Hence in the BS solution, the singularity that occurs when $f + g$=0 corresponds to a Cauchy horizon. This solution is conformally flat in the interaction region; one of the components of the Weyl tensor

$$C_{0202} = -\frac{wq}{2rp(rw + pq)^2(f + g)}[\frac{f}{r^2p^2} f_u^2 + f_{uu}] \tag{7}$$

shows that this solution possesses quasiregular singularities at the null boundaries [4]. Finally, the invariant

$$F_{\alpha\beta}F^{\alpha\beta} = -2\gamma^2 ab \tag{8}$$

which is a finite quantity in the interaction region. The BS solution in the interaction region is diffeomorphic to the Bertotti -Robinson space-time [1, 4].

III. N- DIMENSIONAL EINSTEIN-MAXWELL EQUATIONS

Let M be a $N = 2 + n$ dimensional manifold with a metric

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\, dx^\alpha\, dx^\beta \\ &= g_{ab}(x^c) dx^a\ dx^b + H_{AB}(x^c) dy^A\ dy^B \end{aligned} \tag{9}$$

where $x^\alpha = (x^a,\, y^A)$, x^a denote the local coordinates on a 2-dimensional manifold and y^A denote the local coordinates on n-dimensional manifold, thus $a, b = 1, 2$, $A, B = 1, 2, ..., n$. The Christoffel symbols of the metric $g_{\alpha\beta}$ can be calcullated to give

$$\Gamma^A_{Ba} = \frac{1}{2} H^{AD}\ \ H_{DB,a}, \quad \Gamma^a_{AB} = -\frac{1}{2} g^{ab}\ \ H_{AB,b}, \tag{10}$$

$$\Gamma^A_{BD} = \Gamma^A_{ab} = \Gamma^a_{Ab} = 0, \bar{\Gamma}^a_{bc} = \Gamma^a_{bc} \tag{11}$$

where the Γ^a_{bc} are the Christoffel symbols of the 2-dimensional metric g_{ab}.
The components of the Riemann tensor are given by

$$R^\alpha_{\beta\gamma\sigma} = \Gamma^\alpha_{\beta\sigma,\gamma} - \Gamma^\alpha_{\beta\gamma,\sigma} + \Gamma^\alpha_{\rho\gamma}\, \Gamma^\rho_{\beta\sigma} - \Gamma^\alpha_{\rho\sigma}\, \Gamma^\rho_{\beta\gamma}. \tag{12}$$

The components of the Ricci tensor are

$$\begin{aligned} \mathcal{R}_{ab} &= R^\alpha_{a\alpha b} = R_{ab} + \frac{1}{4} tr(\partial_a H^{-1} \partial_b H) \qquad (13) \\ &- \bigtriangledown_a \bigtriangledown_b\, log\sqrt{detH}, \\ \mathcal{R}_{AB} &= -\frac{1}{2}(g^{ab} H_{AB,b})_{,a} - \frac{1}{2} g^{ab} H_{AB,b} [\frac{(\sqrt{d}etg)_{,a}}{\sqrt{d}etg} \\ &+ \frac{(\sqrt{d}etH)_{,a}}{\sqrt{d}etH}] + \frac{1}{2} g^{ab} H_{EA,b} H^{ED} H_{DB,a}, \qquad (14) \\ \mathcal{R}_{aA} &= 0, \qquad (15) \end{aligned}$$

where R_{ab} is the Ricci tensor of the 2-dimensional metric g_{ab}. The Maxwell potential 1-form $\mathcal{A}$ is

$$\mathcal{A} = A_A dy^A. \tag{16}$$

The components of the electromagnetic field

$$\mathcal{F} = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta \tag{17}$$

are

$$F_{aA} = A_{A,a}, \qquad F_{ab} = 0, \qquad F_{AB} = 0, \tag{18}$$

The components of the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}[g^{\alpha\beta} F_{\mu\beta} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}] \tag{19}$$

are

$$\begin{aligned} T_{ab} &= \frac{1}{4\pi}[H^{AB} F_{aA} F_{bB} - \frac{1}{2} g_{ab} F^2], \\ T_{AB} &= \frac{1}{4\pi}[g^{ab} F_{aA} F_{bB} - \frac{1}{2} H_{AB} F^2], \\ T_{aA} &= 0, \end{aligned} \tag{20}$$

where $F^2 = F_{aD}F^{aD}$. Then the Einstein field equations are

$$R_{\mu\nu} = \kappa[T_{\mu\nu} + \frac{1}{2-N} g_{\mu\nu} T] \tag{21}$$

where the trace of the energy momentum tensor T is

$$T = \frac{1}{8\pi}(2-n)F^2. \tag{22}$$

The Einstein-Maxwell equations are:

$$\begin{aligned} & R_{ab} + \frac{1}{4} tr(\partial_a H^{-1} \partial_b H) - \bigtriangledown_a \bigtriangledown_b \log\sqrt{\det H} \\ &= \frac{\kappa}{4\pi} H^{AB} F_{aA} F_{bB} - \frac{\kappa}{4\pi n} g_{ab} F^2, \qquad (23) \\ & \partial_a[\sqrt{\det H\, g}\ g^{ab} H^{AS} \partial_b H_{AB}] \\ &= -\frac{\kappa}{2\pi}\sqrt{\det H\, g}\ [H^{AS} g^{ab} F_{Aa} F_{Bb} - \frac{\delta^S_B}{n} F^2] \qquad (24) \end{aligned}$$

and

$$\partial_a[\sqrt{\det H\, g}\ F^{Aa}] = 0, \tag{25}$$

where $\bigtriangledown$ is the covariant differentiation with respect to the connection Γ^a_{bc} (or with respect to metric g_{ab}). We may rewrite the 2-dimensional metric as

$$g_{ab} = e^{-M}\, \eta_{ab}, \tag{26}$$

where η is the metric of flat 2-geometry with arbitrary signature (0 or ± 2) the function M depends on the local coordinates x^a. The corresponding Ricci tensor and the Christoffel symbols are

$$\begin{aligned} R_{ab} &= \frac{1}{2}(\bigtriangledown^2_\eta M)\, \eta_{ab}, \\ \Gamma^c_{ab} &= \frac{1}{2}[-M_{,b}\delta^c_a - M_{,a}\delta^c_b + M_{,d}\eta^{cd}\eta_{ab}]. \end{aligned} \tag{27}$$

IV. HIGHER DIMENSIONAL BELL-SZEKERES METRIC

In this section we give the higher dimensional colliding exact plane wave metric generalizing the BS metric. For this purpose let H be a diagonal matrix

$$H = e^{-U} h \tag{28}$$

where

$$h = \begin{pmatrix} e^{V_1} & & & \\ & \ddots & & \bigcirc \\ \bigcirc & & \ddots & \\ & & & e^{V_n} \end{pmatrix}$$

with $\det h = 1$ i.e., $\sum_{k=1}^{n-1} V_k + V_n = 0$.

Now taking the signature of flat-space metric with null coordinates

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x^1 = u,\, x^2 = v,$$

and

$$A_A = (0, ..., A)$$

the Einstein-Maxwell equations become

$$\begin{aligned}
& 2U_{uv} - nU_u U_v = 0, && (29)\\
- & \frac{n}{2} U_u V_{kv} - \frac{n}{2} U_v V_{ku} + 2V_{kuv} && \\
= & \frac{\kappa}{\pi n} e^{U - V_n} A_u A_v, && (30)\\
- & \frac{n}{2} U_u V_{nv} - \frac{n}{2} U_v V_{nu} + 2V_{nuv} && \\
& = \frac{\kappa(1-n)}{\pi n} e^{U - V_n} A_u A_v, && (31)\\
& (\frac{n-2}{2})(U_u A_v + U_v A_u) + V_{nu} A_v + V_{nv} A_u && \\
& = 2A_{uv}, && (32)\\
- & \frac{n}{2} U_u^2 - \frac{1}{2}\sum_{k=1}^{n-1} (V_{ku})^2 - \frac{1}{2}(V_{nu})^2 + nU_{uu} + nM_u U_u && \\
= & \frac{\kappa}{2\pi} e^{U - V_n} (A_u)^2, && (33)\\
- & \frac{n}{2} U_v^2 - \frac{1}{2}\sum_{k=1}^{n-1} (V_{kv})^2 - \frac{1}{2}(V_{nv})^2 + nU_{vv} + nM_v U_v && \\
= & \frac{\kappa}{2\pi} e^{U - V_n} (A_v)^2, && (34)\\
& 2M_{uv} - \frac{n}{2} U_u U_v - \frac{1}{2}\sum_{k=1}^{n-1} V_{kv} V_{ku} - \frac{1}{2} V_{nv} V_{nu} && \\
+ & nU_{uv} = \frac{\kappa}{2\pi n}(2-n) e^{U - V_n} A_u A_v, && (35)
\end{aligned}$$

where $k = 1, ..., n-1$. Note that the last equation is not independent. It can be obtained from the other equations. The most general solution to Eq. (29) is given by

$$U = -\frac{2}{n} \log[f(u) + g(v)] \tag{36}$$

in terms of two arbitrary functions f and g. Now changing variables (u, v) to (f, g) the remaining field equations become

$$\begin{aligned}
& -\frac{n}{2} U_f V_{kg} - \frac{n}{2} U_g V_{kf} + 2V_{kfg} = \frac{\kappa}{\pi n} e^{U - V_n} A_f A_g, && (37)\\
- & \frac{n}{2} U_f V_{ng} - \frac{n}{2} U_g V_{nf} + 2V_{nfg} = \frac{\kappa(1-n)}{\pi n} e^{U - V_n} && \\
& A_f A_g, && (38)\\
& (\frac{n-2}{2})(U_f A_g + U_g A_f) + V_{nf} A_g + V_{ng} A_f && \\
= & 2A_{fg}, && (39)\\
& M_u = -\frac{f_{uu}}{f_u} + \frac{(n-1)}{n}\frac{f_u}{f+g} && \\
- & \frac{(f+g)}{4f_u}[\sum_{k=1}^{n-1} (V_{ku})^2 + (V_{nu})^2 + \frac{\kappa}{\pi} e^{U - V_n} (A_u)^2], && (40)\\
& M_v = -\frac{g_{vv}}{g_v} + \frac{(n-1)}{n}\frac{g_v}{f+g} && \\
- & \frac{(f+g)}{4g_v}[\sum_{k=1}^{n-1} (V_{kv})^2 + (V_{nv})^2 + \frac{\kappa}{\pi} e^{U - V_n} (A_v)^2] && (41)
\end{aligned}$$

Eqs. (37), (38) and (39) are integrability conditions for Eqs. (40) and (41). An exact solution to the above Eqs. (37), (38), and (39) is

$$\begin{aligned}
V_k &= \alpha_k \log(rw - pq) + \beta_k \log(rw + pq), && (42)\\
V_n &= -\alpha \log(rw - pq) - \beta \log(rw + pq), && (43)\\
A &= \gamma(pw - rq) && (44)
\end{aligned}$$

with

$$\alpha_k - \beta_k = \frac{\kappa \gamma^2}{2\pi n} \tag{45}$$

for all $k = 1, ..., n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k = \alpha = \frac{2}{n}, \quad \sum_{k=1}^{n-1} \beta_k = \beta = \frac{2}{n} - 2. \tag{46}$$

Then we may obtain the value of γ as $\gamma^2 = \frac{4n\pi}{\kappa(n-1)}$.

It is convenient to put Eqs. (40) and (41) in the following form [1]

$$e^{-M} = \frac{f_u g_v}{(f+g)^{(n-1)/n}} e^{-S} \tag{47}$$

where S satisfies

$$S_f = -\frac{(f+g)}{4}[\sum_{k=1}^{n-1}(V_{kf})^2 + (V_{nf})^2 \qquad (48)$$
$$+ \frac{\kappa}{\pi}e^{U-V_n}(A_f)^2],$$
$$S_g = -\frac{(f+g)}{4}[\sum_{k=1}^{n-1}(V_{kg})^2 + (V_{ng})^2 \qquad (49)$$
$$+ \frac{\kappa}{\pi}e^{U-V_n}(A_g)^2]$$

Therefore we may write the metric function M as

$$M = -\log(cf_u g_v) + [\frac{n-1}{n} - \frac{4+n^2m_2}{4n^2}]\log(f+g) + (\frac{n}{4(n-1)})\log(\frac{1}{2}-f)$$
$$+(\frac{n}{4(n-1)})\log(\frac{1}{2}+f) + (\frac{n}{4(n-1)})\log(\frac{1}{2}-g) + (\frac{n}{4(n-1)})\log(\frac{1}{2}+g)$$
$$+\frac{1}{8n}[8-4n+n(m_1-m_2)]\log(1+4fg+4prwq) \qquad (50)$$

where c is a constant and

$$\sum_{k=1}^{n-1}\alpha_k^2 = m_1, \quad \sum_{k=1}^{n-1}\beta_k^2 = m_2. \qquad (51)$$

m_1 and m_2, using Eq. (45), satisfy

$$m_1 + m_2 - 2m_3 = \frac{4}{n-1}, \quad m_1 - m_2 = \frac{4(2-n)}{n(n-1)} \qquad (52)$$

with

$$\sum_{k=1}^{n-1}\alpha_k\beta_k = m_3$$

.

The metric function e^{-M} must be continuos across the null boundaries. To make it so we assume that the functions f and g take the form

$$f = \frac{1}{2} - \sin^{n_1} P, \qquad g = \frac{1}{2} - \sin^{n_2} Q. \qquad (53)$$

Then the metric function e^{-M} is continuous across the boundaries if

$$n_1 = n_2 = \frac{4(n-1)}{3n-4}. \qquad (54)$$

Therefore, the metric function e^{-M} reads

$$e^{-M} = \frac{(1+4fg+4pqrw)^{k_1}(1-(\frac{1}{2}-f)^{2/n_1})^{1/2}(1-(\frac{1}{2}-g)^{2/n_1})^{1/2}}{(\frac{1}{2}+f)^{1-\frac{1}{n_1}}(\frac{1}{2}+g)^{1-\frac{1}{n_1}}(f+g)^{k_2}}, \qquad (55)$$

where

$$k_1 = \frac{n-2}{2(n-1)}, \qquad k_2 = \frac{n-1}{n} - \frac{1}{4n^2}(4+n^2m_1). \qquad (56)$$

It may thus be observed that the constant n_1 $(= n_2)$ is restricted to the range satisfying

$$2 \geq n_1 = n_2 > \frac{4}{3}. \qquad (57)$$

It is also appropriate to choose $c = \frac{1}{n_1^2}$.

The spacetime line element generalizing the BS metric in $N = 2+n$ dimensions is

$$ds^2 = 2e^{-M}du\ dv + e^{-U}(e^{V_1}dx_1^2 + ... + e^{V_n}dx_n^2), \quad (58)$$

where the metric functions are given in Eqs. (42-43) and (55). Because of Eq. (57) the metric we have found is C^1 for $n > 2$ across the null boundaries. In spite of this fact, the Ricci tensor is regular across the null boundaries

due to the Einstein field equations. The above solution reduces to the well known BS solution for $n = 2$.

We now discuss the nature of the spacetime singularities. We study the behavior of the metric functions U, V_k, V_n and M as $f + g$ tends to zero. In the BS solution the collision of the two shock electromagnetic plane waves generates impulsive gravitational waves along the null boundaries. It is shown that, apart from the impulsive waves themselves, by virtue of Eq. (6) the BS solution has no curvature singularities and the only singularites are of the quasiregular type [4]. The curvature invariant Eq. (5) for $n > 2$ is

$$I \sim e^{2M} \frac{(f_u g_v)^2}{(f+g)^4}. \tag{59}$$

as $f + g \to 0$. Using M from Eq. (55) we find

$$I \sim (f_u\, g_v)^2\, (f+g)^{2k_2-4} \tag{60}$$

as $f + g \to 0$. It is obvious that spacetimes possess curvature singularities when $k_2 < 2$ and their strength depend on n and m_1.

We also investigate the singularity structure of spacetimes in the context of the Maxwell invariants; one of the invariants is

$$F_{\alpha\beta}F^{\alpha\beta} = -\frac{\gamma^2 n_1^2}{2^{k_1+1}} (rwpq)^{1-\frac{2}{n_1}} (rw+pq)^{-2k_1} \times (f+g)^{k_2} P_u Q_v. \tag{61}$$

which has singularities for $n > 2$ for the negative values of k_2.

We finally examine the Weyl tensor to see whether our spacetime is conformally flat. One of the components of the Weyl tensor in region II for our space-times is

$$\begin{aligned} C_{0n0n} &= \frac{f_u^2}{8(\frac{1}{2}+f)}[-m_1 + \frac{2}{n(n-1)} \\ &+ \frac{2n}{(n-1)}(\alpha+\beta)^3 + \frac{(1-n)}{n}(\alpha+\beta) \\ &+ 2(\alpha+\beta)\frac{(\frac{1}{2}-f)^{-1+2/n_1}(\frac{1}{2}+f)}{(1-(\frac{1}{2}-f)^{2/n_1})}] \\ &+ \frac{(\alpha+\beta)}{4} f_{uu}. \end{aligned} \tag{62}$$

It can be seen that it vanishes only for $n = 2$. Therefore, the higher dimensional extensions of BS metric are not conformally flat.

V. CONCLUSION

In this paper, we give a higher dimensional generalization of BS metric which describes the collision of pure electromagnetic plane waves with collinear polarization in all spacetime dimensions. The solution has two free parameters; the spacetime dimension $N(= 2+n)$ and an arbitrary real number m_1. We show that these spacetimes, unlike BS metric, are not conformally flat. We find that, even though purely electromagnetic plane wave collision in four dimensional spacetime possesses no curvature singularities, in higher dimensions there exist curvature singularities whose nature depend on the real number m_1 and the spacetime dimension.

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[1] J.B, Griffiths, *Colliding Plane Waves in General Relativity*(Clarendon Press,Oxford,1991).
[2] P. Bell and P. Szekeres, Gen. Rel. Grav. **5**,275 (1974).
[3] R. A . Matzner and F.J. Tipler , Phys.Rev. D **292**,1575 (1984).
[4] C. J. S. Clarke and S. A. Hayward, Class. Quantum. Grav. **6**, 615 (1989).
[5] T. M. Helliwell and D. A. Konkowski, Class. Quantum. Grav. **16**, 2709 (1999).
[6] M. Halil, J. Math. Phys. **20**,120 (1979).
[7] M. Gürses and M. Halil, Lett. Nuovo Cimento. **34**, 588 (1982).
[8] J.B, Griffiths, Phys. Lett. A **54**,269 (1975).
[9] S. Chandrasekhar and B. C. Xanthopoulos, Proc. Roy. Soc. A **410**,311 (1987).
[10] M. Gürses and E. Sermutlu, Phys.Rev. D **52**,809 (1995).
[11] M. Halil and I. Sakalli, Class. Quantum. Grav. **20**,1417 (2003).
[12] M. Gürses and A. Karasu, Class. Quantum. Grav. **18**,509 (2001).
[13] M. Gürses , E. O. Kahya, and A. Karasu Phys. Rev. D **66**,024029 (2002).